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Partial Differential Equations

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- Introduction
- Formation of partial differential equations
- Solution of Differential Equations

1.Linear PDE

2. Non-Linear PDE

Partial derivatives

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y(1)$$

$$\frac{\partial}{\partial x}(xy) = y(1) = y$$

$$\frac{\partial}{\partial y}(xy) = x(1) = x$$

$$\frac{\partial}{\partial x}(x + y) = (1) + 0 = 1$$

$$\frac{\partial}{\partial x}(Sin(xy)) = Cos(xy) * (y * 1) = yCos(xy)$$

$$\frac{\partial}{\partial y}(Sin(xy)) = Cos(xy) * (x * 1) = xCos(xy)$$

$$\frac{\partial}{\partial x}(e^{xy}) = (e^{xy})(y) = y(e^{xy})$$

$$\frac{\partial}{\partial y}(e^{xy}) = (e^{xy})(x) = x(e^{xy})$$

Partial derivatives

$$f(x, y) = x^2 - y^2 + 2xy$$
$$\frac{\partial}{\partial x}(f) = 2x + 2y$$
$$\frac{\partial}{\partial y}(f) = -2y + 2x$$
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x + 2y) = 2$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-2y + 2x) = -2$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(-2y + 2x) = 2$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2y + 2x) = 2$$
$$\therefore \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Partial Differential equation:

If an equation involving a dependent variables and its derivatives with respect to two or more independent variables then the equation is said to be partial differential equation.

Example:

1. $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = xyt$$

Here, Z is the **dependent variable** that depends on 3 variables x , y , t.

x , y , t are called the **independent variables**.

2. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

3. $$\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^3 = u$$

ORDER:

The order of a partial differential equation if the order of the highest partial derivative occurring in the equation.

DEGREE:

The degree of a partial differential equation is the greatest exponent of the highest order.

EXAMPLE:

1. $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+t\frac{\partial z}{\partial t}=xyt$ Order:1 Degree:1

2. $(\frac{\partial^2 z}{\partial x^2})^2+(\frac{\partial^2 z}{\partial y^2})^2=2z$ Order:2 Degree:2

3. $(\frac{\partial^3 z}{\partial x^3})^2+(\frac{\partial^3 z}{\partial y^3})^2=2z$ Order:3 Degree:2

NOTATIONS:

Throughout this chapter we use the following notations; z will be taken as a dependent variable which depends on two independent variables x and y so that z = f(x; y).

We write

$$\frac{\partial Z}{\partial x} = p; \qquad \frac{\partial Z}{\partial y} = q$$

$$\frac{\partial^2 z}{\partial x^2} = r; \qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = s; \qquad \frac{\partial^2 z}{\partial y^2} = t$$

Formation of Partial Differential Equation :

1. Formation of partial differential equation by elimination of arbitrary constants.

2. Formation of partial differential equation by elimination of arbitrary functions

elimination of arbitrary constants:

Let $f(x; y; z; a; b) = 0$.
Be an equation which contains two arbitrary constants `a' and `b’.

Now, to eliminate these two constants, Partially differentiating with respect to `x' and `y' we get two more equations. Eliminating a and b from these three equations, we get $\emptyset(x; y; z; p; q) = 0$ which is a partial differential equation of order 1.

In this case the number of arbitrary constants to be eliminated is equal to the number of independent variables and we obtain a first order partial differential equation.

If the number of arbitrary constants to be eliminated is more than the number of independent variables, we get partial differential equations of second or higher order.

2. Formation of partial differential equation by elimination of arbitrary functions:

The elimination of one arbitrary function from a given relation gives a partial differential equation of first order while elimination of two arbitrary function from a given relation gives a second or higher order partial differential equation.

Formation of Partial Differential Equation

By eliminating of arbitrary constants

Problems

Form PDE by eliminating arbitrary constants from $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Sol: we have,

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \rightarrow (1)$$

Differentiating (1) w.r.t to x and y, we get

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \Rightarrow p = \frac{x}{a^2} \Rightarrow a^2 = \frac{x}{p} \rightarrow (2)$$

$$2\frac{\partial z}{\partial y}=\frac{2y}{b^2}\Rightarrow q=\frac{y}{b^2}\Rightarrow b^2=\frac{y}{q}\rightarrow(3)$$

substituting (2) &(3) in (1) we get

$$2z=x^2\frac{p}{x}+y^2\frac{q}{y}$$

$$\Rightarrow 2z=xp+yq$$

which is our required PDE

EXAMPLE:

2. Form the partial differential equation by eliminating the arbitrary constants a and b from log(az-1)= x + a y+ b

SOLUTION:

Given equation is log(az-1)=x + a y + b ------>(1)

Differentiating (1) w. r . To x,

$$\frac{1}{(az-1)} a\left(\frac{\partial Z}{\partial x}\right)=1$$

$$=>\frac{1}{(az-1)} ap=1$$

$$ap = az - 1 \quad ----(1)$$

Differentiating (1) w. r . To y,

$$\frac{1}{(az-1)}a\left(\frac{\partial z}{\partial y}\right)=a$$

$$=>\frac{1}{(az-1)}aq=a$$

$$q = az - 1 \quad ----(2)$$

From e q (2), a p= a z-1

=>a p- a z=-1

=>-1=a(p-z)

=>-1= $\frac{q+1}{z}$(p-z) (from (3))

=>-z=(q+1)(p-z)

$$\therefore z = (z-p)(q+1)$$

3. Form the partial differential equation by eliminating a and b from

$$(x-a)^2+(y-b)^2=(z)^2(Cot)^2\alpha, where\ \alpha\ is\ a\ Constant$$

SOLUTION:

Given equation is $(x-a)^2+(y-b)^2=(z)^2(Cot)^2\alpha$ ------>(1)

Differentiating (1) w r to x,

2(x-a)=2zp $(Cot)^2\alpha$

=>(x-a)=z p $(Cot)^2\alpha$ ----->(2)

Differentiating (1) w r to y,

2(y-b)=2 z q $(Cot)^2\alpha$

=>(y-b)=z q $(Cot)^2\alpha$------>(3)

From (1),(2) and (3),

$$(z\,p\,(Cot)^2\alpha)^2+(z\,q\,(Cot)^2\alpha)^2= z^2Cot^2\alpha$$

$$\Rightarrow z^2p^2Cot^4\alpha + z^2q^2Cot^4\alpha = z^2Cot^2\alpha$$

$$\Rightarrow (p^2 + q^2)Cot^2\alpha=1$$

EXAMPLE:

4. Form the partial differential equation by eliminating the arbitrary constants a and b from$z = axe^y + \frac{1}{2}a^2e^{2y} + b$

SOLUTION:

Given Differential equation is $z = axe^y + \frac{1}{2}a^2e^{2y} + b$ ------>(1)

Differentiating (1) w r to x,

We get p= ae^y------>(2)

Differentiating (1) w r to y,

We get q= $axe^y + \frac{1}{2}a^2e^{2y}2$

$= axe^y + a^2e^{2y}$

=>q=x(ae^y)+(ae^y)^2

=>q=x p+ p^2 (from (2))

5. Form the partial differential equation of all spheres whose radii are the same.

<u>SOLUTION :</u> Let us consider a sphere equation with centre(a , b , c) and radius "r" units.

$(x-a)^2+(y-b)^2+(z-c)^2=(R)^2$--->(1)

Differentiating e q (1) w r to x,

We get 2(x-a)+2(z-c)p=0

=>(x-a)+(z-c)p=0----->(2)

Differentiating (1) w r to y,

We get 2(y-b)+2(z-c)q=0

=>(y-b)+(z-c)q=0----->(3)

From (2), (x-a)+(z-c)p=0

Again differentiating (2) w r to x, We get

1+(z-c)$\frac{\partial p}{\partial x} + p(p-0) = 0$

1+(z-c)r+ p^2=0

=>(z-c)r=-1- p^2

=>(z-c)=$\frac{-1- p^2}{r}$ ------>(4)

From (3), (y-b)+(z-c)q=0

Again differentiating (3) w r to y, We get

$1+(z-c)\frac{\partial q}{\partial y} + q(q-0) = 0$

$=>1+(z-c)t+ q^2=0$

$=>1+ \left(\frac{-1- p^2}{r}\right) t + q^2 = 0$ (from (4))

$=>r+(-1- p^2)t + r\, q^2 = 0$

This is a second order partial differential equation.

1. Find the differential equation of the sphere whose center lies on the z-axis.

<u>Solution:</u> Let (a , b , c) is the center of the sphere.

Given that the center lies on Z-axis.

So , a=0and b=0.

$\therefore$ The equation of the Sphere is $(x)^2+(y)^2+(z-c)^2=(R)^2$--->(1)

Differentiating (1) w.r.to x,

We get 2 x+ 2(z-c)$\left(\frac{\partial Z}{\partial x}\right)$=0

=>x+(z-c)p=0--->(2)

Again differentiating (1) w.r.to y,

We get $2y+2(z-c)\frac{\partial z}{\partial y}=0$

$\Rightarrow y+(z-c)q=0 \rightarrow (3)$

$\therefore$ From (2), $(z-c)p=-x$

$\Rightarrow (z-c)=-x/p$

$\Rightarrow -y/q=-x/p$ (from (3))

$\Rightarrow y\,p=x\,q$

Formation of PDE

By eliminating of arbitrary functions

Problems

1.Form PDE by eliminating arbitrary functions from $z = f(\frac{y}{x})$

Sol: we have,

$$z = f(\frac{y}{x}) \rightarrow (1)$$

Differentiating (1) w.r.t to x and y, we get

$$\frac{\partial z}{\partial x} = f'(\frac{y}{x}) \frac{\partial}{\partial x}(\frac{y}{x}) \Rightarrow p = f'(\frac{y}{x})(\frac{-y}{x^2}) \rightarrow (2)$$

$$\frac{\partial z}{\partial y} = f'(\frac{y}{x})\frac{\partial}{\partial y}(\frac{y}{x}) \Rightarrow q = f'(\frac{y}{x})(\frac{1}{x}) \rightarrow (3)$$

(2) $\div$ (3) ,we get

$$\frac{p}{q} = \frac{-y/x^2}{1/x} = \frac{-y}{x}$$

$$\Rightarrow px + qy = 0$$

which is our required PDE

. Form a partial differential equation by eliminating f from

$z = x^2 + f(\frac{1}{y} + \log x)$

SOLUTION:

Given equation is $z = x^2 + f(\frac{1}{y} + \log x)$ ------>(1)

Differentiating (1) w r to x,

We get p=2 x + $f^I\left(\frac{1}{y} + \log x\right)\left(0 + \frac{1}{x}\right)$

=> p =2 x + $(\frac{1}{x})\, f^I\left(\frac{1}{y} + \log x\right)$

=> P – 2 x = $(\frac{1}{x})\, f^I\left(\frac{1}{y} + \log x\right)$ ------>(2)

Differentiating (1) w r to y,

We get q= 0 + $f^{I}\left(\frac{1}{y}+\log x\right)\left(-\frac{1}{y^2}+0\right)$

=>q = $(-\frac{1}{y^2})\, f^{I}\left(\frac{1}{y}+\log x\right)$ ------>(3)

$$\therefore \frac{(2)}{(3)} \Rightarrow \frac{p-2\,x}{q}=\frac{(\frac{1}{x})\, f^{I}\left(\frac{1}{y}+\log x\right)}{(-\frac{1}{y^2})\, f^{I}\left(\frac{1}{y}+\log x\right)}$$

$$\Rightarrow \frac{p-2\,x}{q}=-\frac{y^2}{x}$$

=>(p-2 x) x =- q (y^2)

Form PDE by eliminating arbitrary functions from

$$f(x^2+y^2, z-xy)=0$$

Sol: we have,

$$f(x^2+y^2, z-xy)=0$$

i.e., $f(x^2+y^2)=z-xy$

Differentiating (1) w.r.t to x and y, we get

$$f'(x^2+y^2)2x=\frac{\partial z}{\partial x}-y \Rightarrow f'(x^2+y^2)2x=p-y \rightarrow (2)$$

$$f'(x^2+y^2)2y=\frac{\partial z}{\partial y}-x \Rightarrow f'(x^2+y^2)2y=q-x \rightarrow (3)$$

(2) $\div$ (3) ,we get

$$\frac{2x}{2y}=\frac{p-y}{q-x}$$

$$\Rightarrow qx-x^2=py-y^2$$

$$\Rightarrow py-qx=y^2-x^2$$

which is our required PDE

Form PDE by eliminating arbitrary function f and g from

z = f(x + a y) +g (x- a y)

<u>SOLUTION:</u>

The given equation is z = f(x + a y) +g (x- a y) ----->(1)

Differentiating (1) w r to x , We get

p= $f^I(x + ay) + g^I(x - a\,y)$ ---->(2)

Differentiating (1) w r to y, we get

q = $f^I(x + ay)$(a)+ $g^I(x - a\,y)$(-a)

=>q=a($f^I(x + ay)$- $g^I(x - a\,y)$) ------>(3)

Again differentiating (2) w r to x, We geta

$$\frac{\partial p}{\partial x} = f^{II}(x + a\,y) + g^{II}(x\ - a\,y)$$

=>r = $f^{II}(x + a\,y) + g^{II}(x\ - a\,y)$ ----->(4)

Again differentiating (3) w r to y, We get

$$\frac{\partial q}{\partial y} = af^{II}(x + a\,y)(0 + a) - a(g^{II}(x\ - a\,y)(0 - a))$$

=> t = $a^2(f^{II}(x + a\,y) + g^{II}(x\ - a\,y))$ ------>(5)

From (4) & (5),

t = a^2 r

Example 6: Form partial differential equations from the solutions

$(i)\quad z = f(x) + e^y g(x)$

$(ii)\quad z = \frac{1}{r}[F(r - at) + F(r + at)]$

Solution: (*i*): Given $z = f(x) + e^y g(x)$

$\therefore \qquad \frac{\partial z}{\partial y} = e^y g(x)$, Keeping $g(x)$ as constant.

and $\qquad \frac{\partial^2 z}{\partial y^2} = e^y g(x)$, (On differentiating again with respect to y)

Thus $\qquad \frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2}$

(*ii*) Given $$z = \frac{1}{r}[F(r-at) + F(r+at)] \quad \ldots(1)$$

$$\frac{\partial z}{\partial t} = \frac{1}{r}\left[F'(r-at)\cdot -a + F'(r+at)\cdot a\right] \quad \ldots(2)$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{a^2}{r}\left[F''(r-at) + F''(r+at)\right] \quad \ldots(3)$$

$$\frac{\partial z}{\partial r} = \frac{1}{r}\left[F'(r-at) + F'(r+at)\right] - \frac{1}{r^2}\left[F(r-at) + F(r+at)\right] \quad \ldots(4)$$

$$\frac{\partial z}{\partial r} = \frac{1}{r}\left[F'(r-at) + F'(r+at)\right] - \frac{z}{r}$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{1}{r}\left[F''(r-at) + F''(r+at)\right] - \frac{1}{r^2}\left[F'(r-at) + F'(r+at)\right] - \frac{1}{r^2}\left[F'(r-at) + F'(r+at)\right] + \frac{2}{r^3}\left[F(r-at) + F(r+at)\right]$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{1}{r}\left[F''(r-at) + F''(r+at)\right] - \frac{2}{r^2}\left[F'(r-at) + F'(r+at)\right] + \frac{2}{r^3}\left[F(r-at) + F(r+at)\right] \quad \ldots(5)$$

On using (1), (3), (4) in (5), we get

$$\frac{\partial^2 z}{\partial r^2} = \frac{1}{a^2}\frac{\partial^2 z}{\partial t^2} - \frac{2}{r}\left[\frac{\partial z}{\partial r} + \frac{z}{r}\right] + \frac{2}{r^2} z$$

$$\frac{\partial^2 z}{\partial r^2} + \frac{2}{r}\frac{\partial z}{\partial r} = \frac{1}{a^2}\frac{\partial^2 z}{\partial t^2} \quad \text{or} \quad \frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial z}{\partial r}\right) = \frac{\partial^2 z}{\partial t^2}$$ is the desired p.d.e.

iii) Form partial differential equation from the relation $z = f_1(x + iy) + f_2(x - iy)$.

Given $$z = f_1(x + iy) + f_2(x - iy) \quad \ldots(1)$$

$$\frac{\partial z}{\partial x} = f_1'(x + iy) + f_2'(x - iy) \quad \ldots(2)$$

$$\frac{\partial z}{\partial y} = i\, f_1'(x + iy) - i\, f_2'(x - iy) \quad \ldots(3)$$

Similarly $$\frac{\partial^2 z}{\partial x^2} = f_1''(x+iy) + f_2''(x-iy) \qquad \ldots(4)$$

$$\frac{\partial^2 z}{\partial y^2} = i^2 f_1''(x+iy) + i^2 f_2''(x-iy) \qquad \ldots(5)$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0; \quad \text{where } i^2 = -1$$

ASSIGNMENT:

1. Form the PDE by eliminating the arbitrary constants from

   (i) z= a x + a^2y^2+b

   (ii) a x+ b y + c z =1.

2. Form the PDE by eliminating the arbitrary function from

   (i) z= f(x- y)

   (ii) $\emptyset\left(xy+z^2,\frac{x}{y}\right)=0$

Different Integrals of Partial Differential Equation

1. Complete Integral (solution)

Let $F(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = F(x, y, z, p, q) = 0......(1)$

be the Partial Differential Equation.

The complete integral of equation (1) is given by $\phi(x, y, z, a, b) = 0.........(2)$

2. Particular solution

A solution obtained by giving particular values to the arbitrary constants in a complete integral is called particular solution .

3.Singular solution

The eliminant of a , b between

$$\phi(x, y, z, a, b) = 0$$

$$\frac{\partial \phi}{\partial a} = 0, \frac{\partial \phi}{\partial b} = 0$$

when it exists , is called singular solution

LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATION:

A differential equation which involves partial derivatives p and q only and no higher order derivatives is called a first order partial differential equation.

If p and q have **degree one** , then the PDE is called a **LINEAR** pde.

If p and q have **degree more than one ,** then the PDE is called a **NON-LINEAR** pde

Examples:

1. $p\,x + q\,y^2 = z$ *Linear PDE*
2. $p\,q + q\,y = z$ *Non − Linear PDE*
3. $p^2 + q^2 = 1$ *Non − Linear PDE*
4. p = q + x y Linear PDE

Solution of Partial Differential Equation

Linear Partial Differential Equation of first order

Lagrange's linear equation:

A Linear Partial Differential Equation of the form

$Pp+Qq=R$

where P,Q,R are functions of x,y.z is called Lagrange's linear equation

To Solve Pp+Qq=R :

1.Write the auxiliary or subsidiary equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

(a) Method of grouping:

In the subsidiary equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$\left(\frac{dx}{P}=\frac{dy}{Q}\right.$ and $\left.\frac{dy}{Q}=\frac{dz}{R}\right)$ or $\left(\frac{dx}{P}=\frac{dz}{R}\right.$ and $\left.\frac{dy}{Q}=\frac{dz}{R}\right)$

if the variables can be separated in any pair of equations,

then we get a

solution of the form u = a and v = b.

Then the general solution is given by

f(u,v)=0 or f(u)=v

(b) Method of multipliers:

Consider $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Identify the multipliers l,m,n not necessarily constants, each ratio equals to

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l dx + m dy + n dz}{lP + mQ + nR}$$

Choose l,m,n So that lP+mQ+nR=0 then ldx+mdy+ndz=0

On integrating we get, $u(x, y, z) = c_1$

Similarly ,by choosing another set of l,m,n we have another independent solution $v(x, y, z) = c_2$

Then the general solution is given by

f(u,v)=0 or f(u)=v

1. Solve the PDE $p\,x + q\,y = z$.

Sol:

Given PDE is $p\,x + q\,y = z$ ---->(1)

This is of the form $p\,P + q\,Q = R$

Here, $P = x$; $Q = y$ and $R = z$.

Now Consider the subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y} \text{ and } \frac{dy}{y} = \frac{dz}{z}$$

Integrating on both sides of these two equations

$\log x = \log y + \log c_1$ and $\log y = \log z + \log c_2$

$\Rightarrow \log x - \log y = \log c_1$ and $\log y - \log z = \log c_2$

$\Rightarrow \log\left(\frac{x}{y}\right) = \log c_1$ and $\log\left(\frac{y}{z}\right) = \log c_2$

$\Rightarrow \left(\frac{x}{y}\right) = c_1$ and $\left(\frac{y}{z}\right) = c_2$

$$\therefore \textit{The solution is } \emptyset\left(\frac{x}{y}, \frac{y}{z}\right) = 0$$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\Rightarrow \frac{dx}{x} = \frac{dz}{z} \text{ and } \frac{dy}{y} = \frac{dz}{z}$$

Integrating on both sides of these two equations

$\log x = \log z + \log c_1$ and $\log y = \log z + \log c_2$

$\Rightarrow \log x - \log z = \log c_1$ and $\log y - \log z = \log c_2$

$\Rightarrow \log \left(\frac{x}{z}\right) = \log c_1$ and $\log \left(\frac{y}{z}\right) = \log c_2$

$\Rightarrow \left(\frac{x}{z}\right) = c_1$ and $\left(\frac{y}{z}\right) = c_2$

$$\therefore \textit{The solution is } \emptyset\left(\frac{x}{z}, \frac{y}{z}\right) = 0$$

2. Solve the PDE p x + q y = 0.

Sol:

Given PDE is p x + q y = 0 ---->(1)

This is of the form p P + q Q = R

Here, P =x ; Q= y and R=0.

Now Consider the subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$\Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{0}$

$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$ and $\frac{dy}{y} = \frac{dz}{0}$

$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$ and $dz=0$

Integrating on both sides of these two equations

$\log x = \log y + \log c_1$ and $z=c_2$

$\Rightarrow \log x - \log y = \log c_1$ and $z=c_2$

$\Rightarrow \log\left(\frac{x}{y}\right) = \log c_1$ and $z=c_2$

$\Rightarrow \left(\frac{x}{y}\right) = c_1$ and $z=c_2$

$$\therefore \textit{The solution is } \emptyset\left(\frac{x}{y}, z\right) = 0$$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{0}$$

$$\Rightarrow \frac{dx}{x} = \frac{dz}{0} \text{ and } \frac{dy}{y} = \frac{dz}{0}$$

$\Rightarrow$ dz = 0 and dz = 0

Integrating on both sides of these two equations

Z= c_1 and Z= c_2

$$\therefore \textit{The solution is } \emptyset(z, z) = 0, \textit{the solution cannot be like this}$$

3. Solve the PDE $\frac{y^2 z}{x}\, p + x\, z\, q = y^2$.

Sol:

Given PDE is $\frac{y^2 z}{x}\, p + x\, z\, q = y^2$

Multiplying both sides with x, we get

$y^2 z\ \ p + x^2 z\ \ q = y^2 x$ ---->(1)

This is of the from p P + q Q = R.

Where P = $y^2 z$; Q = $x^2 z$ and R= y^2x

Now Consider the subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{y^2 z} = \frac{d\,y}{x^2 z} = \frac{d\,z}{y^2 x}$$

$$\Rightarrow \frac{dx}{y^2 z} = \frac{d\,y}{x^2 z} \text{ and } \frac{d\,y}{x^2 z} = \frac{d\,z}{y^2 x}$$

$$\Rightarrow \frac{d\,x}{y^2} = \frac{d\,y}{x^2} \; and \; \frac{d\,y}{xz} = \frac{d\,z}{y^2}$$, which is impossible

$$\Rightarrow \frac{d\,x}{y^2} = \frac{d\,y}{x^2} \text{ and } \frac{dx}{y^2 z} = \frac{d\,z}{y^2 x}$$

$\frac{d\,x}{y^2} = \frac{d\,y}{x^2}$ and $\frac{dx}{y^2 z} = \frac{d\,z}{y^2 x}$

$\Rightarrow x^2\,dx = y^2\,dy \;\; and\; x\,dx = z\,dz$

Integrating on both sides of these two equations

$$\frac{x^3}{3} = \frac{y^3}{3} + C_1 \; and \; \frac{x^2}{2} = \frac{z^2}{2} + C_2$$

$\frac{x^3}{3} - \frac{y^3}{3} = C_1 \; and \; \frac{x^2}{2} - \frac{z^2}{2} = C_2$

The solution is $\emptyset\left(\frac{x^3}{3} - \frac{y^3}{3}, \frac{x^2}{2} - \frac{z^2}{2}\right) = 0$

4.Solve the PDE $p\sqrt{x}+q\sqrt{y}=\sqrt{z}$

Sol: Given,

$$p\sqrt{x}+q\sqrt{y}=\sqrt{z}$$

The auxiliary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

i.e., $$\frac{dx}{\sqrt{x}}=\frac{dy}{\sqrt{y}}=\frac{dz}{\sqrt{z}}$$

Consider, $\dfrac{dx}{\sqrt{x}} = \dfrac{dy}{\sqrt{y}}$

On integrating ,we get

$$2\sqrt{x} = 2\sqrt{y} + c$$

$$\Rightarrow \sqrt{x} - \sqrt{y} = c_1$$

now consider, $\dfrac{dy}{\sqrt{y}} = \dfrac{dz}{\sqrt{z}}$

On integrating ,we get

$$2\sqrt{y} = 2\sqrt{z} + c$$

$$\Rightarrow \sqrt{y} - \sqrt{z} = c_2$$

Hence the general solution is $f(c_1, c_2) = 0$

i.e.

$f(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$, where f is arbitrary

5. Solve the PDE $x^2 p + y^2 q = z^2$

Sol:

The given PDE is $x^2 p + y^2 q = z^2$ ------>(1)

The given PDE is of the form p P + q Q = R

Here , P = x^2 ; Q = y^2 and R = z^2

Now Consider the subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

$\Rightarrow \frac{dx}{x^2} = \frac{dy}{y^2}$ and $\frac{dy}{y^2} = \frac{dz}{z^2}$

$\Rightarrow$ Integrating on both sides of these two equations we get

$\Rightarrow \frac{-1}{x} = \frac{-1}{y} + C_1 and \ \frac{-1}{y} = \frac{-1}{z} + C_2$

$\Rightarrow \frac{1}{y} - \frac{1}{x} = C_1 \ and \ \frac{1}{z} - \frac{1}{y} = C_2$

$$\therefore The\ general\ solution\ is\ \emptyset\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right) = 0$$

6. Solve the PDE (a - x)p + (b - y)q = c – z

Sol:

The given PDE is (a - x)p + (b - y)q = c – z.

This is of the from p P + q Q = R

Here, P= a-x ; Q = b-y and R=c-z.

Now Consider the subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{a-x} = \frac{d\,y}{b-y} = \frac{dz}{c-z}$$

$$\Rightarrow \frac{dx}{a-x} = \frac{d\,y}{b-y} \text{ and } \frac{d\,y}{b-y} = \frac{dz}{c-z}$$

$\Rightarrow$ Integrating both sides of these two equations, we get

- log (a-x)=-log (b-y) + log C_1 and – log (b-y)=-log (c-z) + log C_2

=>log (b-y)-log(a-x)= log C_1 and log(c-z) – log(b-y)= log C_2

=>log $\left(\frac{b-y}{a-x}\right)$ = log C_1 and log $\left(\frac{c-z}{b-y}\right)$ = log C_2

$$\therefore \textit{The generral solution is } \emptyset\left(\frac{b-y}{a-x}, \frac{c-z}{b-y}\right) = 0$$

5.Solve the PDE $p\tan x + q\tan y = \tan z$

Sol: Given,

$$p\tan x + q\tan y = \tan z$$

The auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e., $$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Consider, $$\frac{dx}{\tan x} = \frac{dy}{\tan y} \Rightarrow \cot x dx = \cot y dy$$

On integrating ,we get

$$\log \sin x = \log \sin y + \log c_1$$

$$\Rightarrow \log \frac{\sin x}{\sin y} = \log c_1 \Rightarrow \frac{\sin x}{\sin y} = c_1$$

now consider, $$\frac{dy}{\tan y} = \frac{dz}{\tan z} \Rightarrow \cot y dy = \cot z dz$$

On integrating ,we get

$$\log \sin y = \log \sin z + \log c_2$$

$$\Rightarrow \log \frac{\sin y}{\sin z} = \log c_2 \Rightarrow \frac{\sin y}{\sin z} = c_2$$

Hence the general solution is $f(c_1, c_2) = 0$

i.e

$$f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$,where f is arbitrary

3.Solve the PDE $px - qy = y^2 - x^2$

Sol: Given,

$$px - qy = y^2 - x^2$$

The auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e.,

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2 - x^2}$$

Consider, $\dfrac{dx}{x} = \dfrac{dy}{-y}$

On integrating ,we get

$$\log x = -\log y + \log c_1$$

$$\Rightarrow \log xy = \log c_1 \Rightarrow xy = c_1$$

Now taking l=x,m=y,n=1 as multipliers, we get

$$\frac{xdx + ydy + dz}{x^2 - y^2 + y^2 - x^2} = \frac{xdx + ydy + dz}{0}$$

$$\therefore xdx + ydy + dz = 0$$

on integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}+z=c \Rightarrow x^2+y^2+2z=c_2$$

Hence the general solution is $f(c_1, c_2)=0$

i.e.,

$f(xy, x^2+y^2+2z)=0$,where f is arbitrary

.Solve the PDE $x^2(y-z)p+y^2(x-z)q=z^2(x-y)$

Sol: Given,

$$x^2(y-z)p+y^2(x-z)q=z^2(x-y)$$

The auxiliary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

i.e.,

$$\frac{dx}{x^2(y-z)}=\frac{dy}{y^2(z-x)}=\frac{dz}{z^2(x-y)}\rightarrow(1)$$

Now (1) can be written as

$$\frac{\frac{1}{x^2}dx}{(y-z)} = \frac{\frac{1}{y^2}dy}{(z-x)} = \frac{\frac{1}{z^2}dz}{(x-y)}$$

Now taking l=1,m=1,n=1 as multipliers, we get

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{y-z+z-x+x-y} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\therefore \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

on integrating, we get $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1$

Also (1) can be written as

$$\frac{\frac{1}{x}dx}{x(y-z)} = \frac{\frac{1}{y}dy}{y(z-x)} = \frac{\frac{1}{z}dz}{z(x-y)}$$

Now taking l=1,m=1,n=1 as multipliers, we get

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{xy - xz + yz - yx + zx - zy} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

on integrating, we get

$$\log x + \log y + \log z = \log c_2 \Rightarrow xyz = c_2$$

Hence the general solution is $f(c_1, c_2) = 0$

i.e.,

$f(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz) = 0$,where f is arbitrary

EXAMPLE: 1. Solve (y - z)p + (z - x)q = x - y.

Sol: Given equation is (y - z)p + (z - x)q = x - y.

This is of the form p P + q Q = R

Here P = y-z , Q = z-x and R = x-y

Now Consider the subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$=> \frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$$

$\Rightarrow \frac{dx}{y-z} = \frac{dy}{z-x}$ and $\frac{dy}{z-x} = \frac{dz}{x-y}$ (or) $\frac{dx}{y-z} = \frac{dz}{x-y}$ and $\frac{dy}{z-x} = \frac{dz}{x-y}$

?????

So, here we can use the method of multipliers.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{l\,P + m\,Q + n\,R} \text{------>(1)}$$

For this we have to choose l , m , n such that l P + m Q + n R = 0

=> ( ) (y-z) + ( ) (z-x) + ( ) (x-y) = 0 what are l, m, n ?

(i)

$(1)(y-z) + (1)(z-x) + (1)(x-y) = y - z + z - x + x - y = 0$

So, $(l, m, n) = (1, 1, 1)$

(ii)

$x(y-z) + y(z-x) + z(x-y)$

$= xy - xz + yz - yx + zx - zy$

$= 0$

So, $(l, m, n) = (x, y, z)$

Now from (1) and (i) ,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{1\,dx + 1\,dy + 1\,dz}{0}$$

$\Rightarrow$dx + dy +dz =0

$\Rightarrow$Integrating on both sides,

$\Rightarrow$X + y + z = a ----→ (2)

Now from (1) and (ii) ,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\Rightarrow$X dx + y dy + z dz=0

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = b \longrightarrow (3)$

$\therefore$ The general solution is Ø (X + y + z , $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}$) =0.

NOTE:

Some times it is not possible to solve a Lagrange's linear equation by either Method grouping or Method multipliers. In such cases we add / subtract the terms of the subsidiary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ to get the solution.

. Solve: $(x^2 - y^2 - z^2)p + 2\,x\,y\,q - 2\,x\,z = 0$.

The given Pde is $(x^2 - y^2 - z^2)p + 2\,x\,y\,q - 2\,x\,z = 0$

$\Rightarrow (x^2 - y^2 - z^2)p + 2\,x\,y\,q = 2\,x\,z$

This is of the form p P + q Q = R

Here, $P = (x^2 - y^2 - z^2)$

$Q = 2\,x\,y$

$R = 2\,x\,z$

We observe that there are no l, m, n values satisfying

l P + m Q+ n R =0

The subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$=> \frac{d\,x}{(x^2 - y^2 - z^2)} = \frac{dy}{2\,x\,y} = \frac{d\,z}{2\,x\,z}$

$(A) \qquad (B) \qquad (C)$

From (B) and (C),

$$\frac{d\,y}{2\,x\,y} = \frac{d\,z}{2\,x\,z}$$

$=> \frac{dy}{y} = \frac{dz}{z}$

Integrating on both sides, $\log y = \log z + \log C_1$

$\Rightarrow$Log y- log z = log C_1

$\Rightarrow$Log $(\frac{y}{z}) = \log C_1$

$\Rightarrow \frac{y}{z} = C_1$

Now consider x (A) + y (B) +z (C) = (B)

$$\Rightarrow \frac{d\,y}{2\,x\,y} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2 - y^2 - z^2) + y(2xy) + z(2xz)}$$

$$\Rightarrow \frac{d\,y}{2\,x\,y} = \frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2}$$

$$\Rightarrow \frac{d\,y}{2\,x\,y} = \frac{x\,dx + y\,dy + z\,dz}{x^3 + xy^2 + xz^2}$$

$$\Rightarrow \frac{d\,y}{2\,x\,y} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)}$$

$$\Rightarrow \frac{d\,y}{2\ \ y} = \frac{x\,dx + y\,dy + z\,dz}{(x^2 + y^2 + z^2)}$$

$$\Rightarrow \frac{d\,y}{y} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{(x^2 + y^2 + z^2)}$$

$$\frac{dy}{y} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{(x^2 + y^2 + z^2)}$$

Integrating on both sides

Log y =log $(x^2 + y^2 + z^2)$ + log C_2

=>Log y - log $(x^2 + y^2 + z^2)$ = log C_2

=>Log ($\frac{y}{x^2+y^2+z^2}$) = log C_2

$\Rightarrow$($\frac{y}{x^2+y^2+z^2}$) = C_2

$\therefore$ The general solution is $\emptyset$ ($\frac{y}{z}$, $\frac{y}{x^2+y^2+z^2}$)=0.

Practice problems

1.Form the PDE by eliminating arbitrary fuction from $z = f(x^2 + y^2)$

2. Solve the PDE $y^2zp + x^2zq = y^2x$

3. Solve the PDE $x(y-z)p + y(x-z)q = z(x-y)$

Non-Linear Partial Differential Equation

A PDE which involves first order partial derivatives p and q with degree higher than one and the products of p and q is called a non-linear PDE

$$p^2 + pq = z$$

There are six types of non-linear PDE of first order

Type-I: Equations of the type f(p,q)=0

Method of solution:

Let the required solution be z=ax+by+c

then $p=\frac{\partial z}{\partial x}=a$ and $q=\frac{\partial z}{\partial y}=b$

substituiting in f(p,q)=0, we get

f(a,b)=0

From this we obtain b in terms of a. let b=Ø(a)

Then the the required solution be z=ax+Ø(a)y+c

Problems

1.Solve $pq = 1$

Sol: Let the solution be

$$z = ax + by + c \longrightarrow (1)$$

$$\therefore\ p = \frac{\partial z}{\partial x} = a, q = \frac{\partial z}{\partial y} = b$$

substitute in the given equation, we get

$$ab = 1 \Longrightarrow b = \frac{1}{a}$$

From (1),

$$z = ax + \frac{y}{a} + c$$ which is the required solution

2. Solve the PDE $\sqrt{p} + \sqrt{q} = 1$

Sol : The given Pde is $\sqrt{p} + \sqrt{q} = 1$----->(1)

This is of the form f(p , q)=0

Let the solution is z = a x + b y + c --->(2)

Differentiating (2) w r to x, y respectively we get

$$\frac{\partial z}{\partial x} = a \text{ and } \frac{\partial z}{\partial y} = b$$

$\Rightarrow$ p= a and q = b

From (1), $\sqrt{a} + \sqrt{b} = 1$

=> $\sqrt{b} = 1 - \sqrt{a}$

=> b =$(1 - \sqrt{a})^2$

$\therefore$ The general solution is z = a x + $(1 - \sqrt{a})^2$ y + c (from (2))

3 . Solve the Pde $p^2 + q^2 = 2p\,q$

Sol: The given Pde is $p^2 + q^2 = 2p\,q$----->(1)

This is of the form f(p , q)=0

Let the solution is z = a x + b y + c --->(2)

Differentiating (2) w r to x, y respectively we get

$$\frac{\partial z}{\partial x} = a \text{ and } \frac{\partial z}{\partial y} = b$$

$\Rightarrow$ p= a and q = b

From (1), $a^2 + b^2 = 2ab$

=> $a^2 + b^2 - 2\,a\,b = 0$

=> $(a - b)^2$=0

=>a =b

$\therefore$ The general solution is z = a x +a y + c (from (2))

4.Solve $p^2 + q^2 = npq$

Sol: Let the solution be $z = ax + by + c \longrightarrow (1)$

$$\therefore\ p = \frac{\partial z}{\partial x} = a, q = \frac{\partial z}{\partial y} = b$$

substitute in the given equation, we get

$$a^2 + b^2 = nab \Rightarrow b^2 - nab + a^2 = 0 \Rightarrow b = \frac{na \pm \sqrt{n^2a^2 - 4a^2}}{2}$$

$$= \frac{a}{2}[n \pm \sqrt{n^2 - 4}]$$

From (1),

$z = ax + \frac{ay}{2}[n \pm \sqrt{n^2 - 4}] + c$ which is the required solution

Type-II: Equations of the type f(z,p,q)=0

Method of solution:

Let the required solution be z=f(x+ay)=f(u) i.e., u= x+ay

then $p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}$ and $q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$

substituiting the values p and q in f(z,a,b)=0, we get

$$f(z, \frac{dz}{du}, a\frac{dz}{du}) = 0$$

The solution of this ordinary DE will give the solution of f(z,p,q)=0

Solve the Pde $Z = p\,q$.

Sol: The given Pde is $Z = p\,q$ ----->(1)

This is of the form $f(z, p, q)=0$.

Let $z = f(x + ay)$ is the solution of (1)

Let $u = x + ay$ so that $z = f(u)$

$\Rightarrow \frac{\partial z}{\partial x} = \frac{dz}{du}(1 + 0)$ and $\frac{\partial z}{\partial y} = \frac{dz}{du}(0 + a(1))$

$\Rightarrow \frac{\partial z}{\partial x} = \frac{dz}{du}$ and $\frac{\partial z}{\partial y} = a\frac{dz}{du}$

$\Rightarrow$ From (1), $z = a\left(\frac{dz}{du}\right)^2$

$\Rightarrow \frac{z}{a} = \left(\frac{dz}{du}\right)^2$

$$\frac{z}{a} = \left(\frac{dz}{du}\right)^2$$

$$\Rightarrow \frac{dz}{du} = \sqrt{\frac{z}{a}}$$

$$\Rightarrow \frac{dz}{\sqrt{z}} = \frac{du}{\sqrt{a}}$$

$$\Rightarrow z^{-\frac{1}{2}}\, dz = \frac{1}{\sqrt{a}}\, du$$

$\Rightarrow$ Integrating on both sides, $\dfrac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \dfrac{1}{\sqrt{a}} u + c$

$$\Rightarrow 2\sqrt{z} = \frac{u}{\sqrt{a}} + c$$

$$\Rightarrow 2\sqrt{za} = u + c\sqrt{a}$$

$2\sqrt{za} = u + c\sqrt{a}$

=> $2\sqrt{za}$ = x +ay + $c\sqrt{a}$ is the required complete integral.

Solve the Pde $p(1 + q) = q z$

Sol: the given Pde is $p(1 + q) = q z$ --->(1)

This is of the form $f(z, p, q)=0$.

Let $z = f(x + ay)$ is the solution of (1)

Let $u= x + ay$ so that $z = f(u)$

$\Rightarrow \frac{\partial z}{\partial x} = \frac{dz}{du}(1 + 0) and \frac{\partial z}{\partial y} = \frac{dz}{du}(0 + a(1))$

$\Rightarrow$ p= $\frac{dz}{du} and q = a\frac{dz}{du}$

$\Rightarrow$ From (1), $\frac{dz}{du}(1 + a\frac{dz}{du}) = a\frac{dz}{du}z$

$\Rightarrow (1 + a\frac{dz}{du})= az$

$$\left(1 + a\frac{dz}{du}\right) = az$$

$$\Rightarrow a\frac{dz}{du} = az - 1$$

$$\Rightarrow \frac{dz}{az-1} = \frac{du}{a}$$

$\Rightarrow$ Integrating on both sides, we get

$$\Rightarrow \frac{\log(az-1)}{a} = \frac{u}{a} + c$$

$\Rightarrow$ Log (az-1)=u +c

$\Rightarrow$ Log (az-1)= x +ay +c is

Solve the Pde 9 (p^2 z + q^2)=4

Sol: The given Pde is 9 (p^2 z + q^2)=4---->(1)

This is of the form f (z , p, q)=0.

Let z = f(x + ay) is the solution of (1)

Let u= x + ay so that z = f (u)

$\Rightarrow \frac{\partial z}{\partial x} = \frac{dz}{du}(1 + 0) and \frac{\partial z}{\partial y} = \frac{dz}{du}(0 + a(1))$

$\Rightarrow$ p= $\frac{dz}{du} andq = a\frac{dz}{du}$

$\Rightarrow$ From (1), 9 {$(\frac{dz}{du})^2$z + a^2 $(\frac{dz}{du})^2$} =4

$\Rightarrow$ {$(\frac{dz}{du})^2 z + a^2 \ (\frac{dz}{du})^2$}=4/9

$$\{(\frac{dz}{du})^2 z + a^2\ (\frac{dz}{du})^2\} = 4/9$$

$$\Rightarrow (\frac{dz}{du})^2 \{\ z + \ a^2\ \} = \frac{4}{9}$$

$$\Rightarrow (\frac{dz}{du})^2 = \frac{4}{9(z + a^2)}$$

$$\Rightarrow \frac{dz}{du} = \frac{2}{3\sqrt{z+a^2}}$$

$$\Rightarrow\ 3\sqrt{z + a^2}\ \mathrm{dz} = 2\ \mathrm{du}$$

$\Rightarrow$ Integrating on both sides,

$$\Rightarrow 3\,\frac{(z+a^2)^{\frac{3}{2}}}{3/2} = u + c$$

$$\Rightarrow 2\ (z + a^2)^{\frac{3}{2}} = x + ay + c$$

Problems

Solve $p^2 + pq = z^2$

Sol: The equation is in the form of f(z,p,q)=0

Let z=f (u) where u=x+ay

$$\therefore p = \frac{dz}{du}, q = a\frac{dz}{du}$$

substitute p and q values in the given equation , we get

$$\left(\frac{dz}{du}\right)^2 + a\left(\frac{dz}{du}\right)^2 = z^2$$

$$\Rightarrow (1+a)\left(\frac{dz}{du}\right)^2 = z^2$$

$$\Rightarrow \frac{dz}{du} = \frac{z}{\sqrt{1+a}}$$

$$\Rightarrow \frac{dz}{z} = \frac{du}{\sqrt{1+a}}$$

on integrating, we get

$$\Rightarrow \log z = \frac{u}{\sqrt{1+a}} + c$$

The required solution is $\log z = \dfrac{x+ay}{\sqrt{1+a}} + c$

Solve $z = p^2 + q^2$

Sol: The equation is in the form of f(z,p,q)=0

Let z=f(u) where u=x+ay

$$\therefore p = \frac{dz}{du}, q = a\frac{dz}{du}$$

substitute p and q values in the given equation , we get

$$z = (\frac{dz}{du})^2 + a^2(\frac{dz}{du})^2 = (1+a^2)(\frac{dz}{du})^2$$

$$\Rightarrow \frac{dz}{du} = \sqrt{\frac{z}{1+a^2}}$$

$$\Rightarrow \frac{dz}{\sqrt{z}} = \frac{du}{\sqrt{1+a^2}}$$

on integrating, we get

$$\Rightarrow 2\sqrt{z} = \frac{u}{\sqrt{1+a^2}} + c$$

The required solution is $2\sqrt{z} = \dfrac{x+ay}{\sqrt{1+a^2}} + c$

Type-III: Equations of the form $f(x,p)=g(y,q)$

Method of solution: Assume $f(x,p)=g(y,q)=k$

solving for p and q, we get

$$p=F(x,k),q=G(y,k)$$

since z is a function of x and y, we have

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy= pdx+qdy=F(x,k)dx+G(y,k)dy$$

on integrating

$$z=\int F(x,k)dx+\int G(y,k)dy+b$$

Which is the required solution

Problems

1.Solve $p+q=\sin x+\sin y$

Sol: Given, $p+q=\sin x+\sin y$

$$\Rightarrow\ p-\sin x=\sin y-q$$

Assume,

$$p-\sin x=\sin y-q=k$$

$$\therefore\ p=\sin x+k, q=\sin y-k$$

we have,

$$dz == pdx+qdy$$

Substituting p and q values, we get

$$dz = (\sin x + k)dx + (\sin y - k)dy$$

on integrating, we get

$$z = -\cos x + kx - \cos y - ky + c$$

$$\Rightarrow z = -(\cos x + \cos y) + k(x - y) + c$$

which is the required solution

. Solve the Pde $p + q = x + y$

Sol; The given Pde is $p + q = x + y$ ----->(1)

This is of the form $f(x, p) = g(y, q)$

From (1),

let $p - x = y - q = K$ (say)

$\Rightarrow p = K + x$ and $q = -K + y$ ------>(2)

We know that

$dz = p\,dx + q\,dy$

$\Rightarrow dz = (K + x)\,dx + (-K + y)\,dy$ (from (2))

$d\,z = (K + x)\,dx + (-K + y)\,dy$

Integrating on both sides ,

we get

$Z = K\,x + \frac{x^2}{2} - K\,y + \frac{y^2}{2} + C$ *is the general solution.*

Solve the Pde $p^2 - x = q^2 - y$

Sol: The given Pde is $p^2 - x = q^2 - y \dashrightarrow (1)$

This is of the form f (x, p)= g (y, q)

From (1),

$p^2 - x = q^2 - y = \mathrm{K}$ (say)

$=> p^2 = K + x \; and \; q^2 = K + y$

$=> p = (K + x)^{\frac{1}{2}} \; and \; q = (K + y)^{\frac{1}{2}}$ ---->(2)

We have dz = p dx + q dy

$=> dz = (K + x)^{\frac{1}{2}} \; dx + (K + y)^{\frac{1}{2}} \, dy$

$$d\,z = (K + x)^{\frac{1}{2}}\; dx + (K + y)^{\frac{1}{2}}\, dy$$

Integrating on both sides,

We get

$$Z = \frac{(K+x)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(K+y)^{\frac{3}{2}}}{\frac{3}{2}} + C$$

(or)

$$\frac{3}{2}\, z = (K + x)^{\frac{3}{2}} + (K + y)^{\frac{3}{2}} + \frac{3}{2} C$$

. Solve the Pde $\frac{x}{p} + \frac{y}{q} = 1$

Sol: The given Pde is $\frac{x}{p} + \frac{y}{q} = 1$ ---->(1)

This is of the form f (x, p)= g(y , q)

From (1),

$\frac{x}{p} = 1 - \frac{y}{q} = \text{K (say)}$

$\Rightarrow \frac{x}{p} = K \ and 1 - \frac{y}{q} = K$

$\Rightarrow \frac{x}{K} = p \ and \ 1 - K = \frac{y}{q}$

$\Rightarrow \frac{x}{K} = p \ and \ q = \frac{y}{1-K}$ --->(2)

We know that

dz = p dx + q dy

$\Rightarrow$ d z = $\frac{x}{K}$ dx + $\frac{y}{1-K}$ dy

$\Rightarrow$ Integrating on both sides,

We get

Z = $\frac{x^2}{2\,K} + \frac{y^2}{2(\,1-K)} + C$ *is the required solution.*

Type-IV(Clairaut's form): Equations of the form z=px+qy+f(p,q)

Method of solution:

Let the required solution be z=ax+by+c

then $p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$

substituiting in p=a and q=b in the given equation, we get

z=ax+by+f(a,b)

which is the required solution

Problems

1.Solve $z = px + qy + p^2q^2$

Sol:Given,

$$z = px + qy + p^2q^2 \longrightarrow (1)$$

It is in the form of z=px+qy+f(p,q)

Let the solution be $z = ax + by + c \longrightarrow (2)$

$$\therefore\ p = \frac{\partial z}{\partial x} = a, q = \frac{\partial z}{\partial y} = b$$

substitute in (1), we get

$$z = ax + by + a^2b^2$$

which is the required solution

Problems

2.Solve $pqz = p^2(qx + p^2) + q^2(py + q^2)$

Sol:Given,

$$pqz = p^2(qx + p^2) + q^2(py + q^2)$$

$$\Rightarrow pqz = p^2q(x + \frac{p^2}{q}) + q^2p(y + \frac{q^2}{p})$$

$$\Rightarrow z = p(x + \frac{p^2}{q}) + q(y + \frac{q^2}{p})$$

$$\Rightarrow z = (px + qy) + \frac{p^3}{q} + \frac{q^3}{p} \rightarrow (1)$$

It is in the form of z=px+qy+f(p,q)

Let the solution be $z = ax + by + c \longrightarrow (2)$

$$\therefore\ p = \frac{\partial z}{\partial x} = a, q = \frac{\partial z}{\partial y} = b$$

substitute in (1), we get

$$z = (ax + by) + \frac{a^3}{b} + \frac{b^3}{a}$$

which is the required solution

Type-V: Equations of the type $f(x^m p, y^n q) = 0$ and $f(x^m p, y^n q, z) = 0$

Method of solution:

The above form can be transformed to f(P,Q)=0 or f(P,Q,z)=0 by following substitution

case (i): when $m \neq 1$ and $n \neq 1$

put $X = x^{1-m}, Y = y^{1-n}$, then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = P(1-m)x^{-m} \text{ where } P = \frac{\partial z}{\partial X}$$

$$\Rightarrow x^m p = P(1-m)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x}=Q(1-n)y^{-n} \quad \text{where } Q=\frac{\partial z}{\partial Y}$$

$$\Rightarrow y^n q=Q(1-n)$$

Then the given equation reduces to f(P,Q)=0 or f(P,Q,z)=0

Case (ii): when m=1 and n=1

put X=logx and Y=logy. Then

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial x}=P\frac{1}{x}\Rightarrow px=P \quad \text{where } P=\frac{\partial z}{\partial X}$$

Similarly, $qy=Q$

Then the given equation reduces to f(P,Q)=0 or f(P,Q,z)=0

Problems

1.Solve $xp + yq = 1$

Sol:Given,

$$xp + yq = 1 \rightarrow (1)$$

This is in the form of $f(x^m p, y^n q) = 0$

Here m=1 and n=1

Put X=logx and Y=logy

now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = P\frac{1}{x} \Rightarrow px = P$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x} = Q\frac{1}{y} \Rightarrow qy = Q$$

Substituting in (1), we get

$$P + Q = 1 \longrightarrow (2)$$

Let z=Ø (u) where u=X+aY be the solution of (2)

Then

$$P = \frac{dz}{du}, Q = a\frac{dz}{du}$$

From (2),

$$\frac{dz}{du}(1+a) = 1 \Longrightarrow \frac{dz}{du} = \frac{1}{1+a}$$

$$\Longrightarrow dz = \frac{1}{1+a}du$$

On integrating, we get

$$z = \frac{1}{1+a} u + c$$

$$\Rightarrow z = \frac{1}{1+a}(X + aY) + c$$

Hence the general solution is

$$z = \frac{1}{1+a}(\log x + a \log y) + c$$

Problems

2.Solve $\dfrac{x^2}{p}+\dfrac{y^2}{q}=z$

Sol:Given,

$$\frac{x^2}{p}+\frac{y^2}{q}=z \Rightarrow (x^{-2}p)^{-1}+(y^{-2}q)^{-1}=z \rightarrow (1)$$

This is in the form of $f(x^m p, y^n q, z)=0$

Here m=-2 and n=-2

Put $X=x^{1-m}=x^{1-(-2)}=x^3 \;\&\; Y=y^{1-m}=y^{1-(-2)}=y^3$

now, $p=\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial X}\dfrac{\partial X}{\partial x}=P3x^2 \Rightarrow x^{-2}p=3P$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x}=Q3y^2 \Rightarrow x^{-2}q=3Q$$

Substituting in (1), we get

$$(3P)^{-1} + (3Q)^{-1} = z \rightarrow (2)$$

It is in the form of f(z,P,Q)=0

Let z=f(u) where u=X+aY be the solution of (2)

Then $P = \frac{dz}{du}, Q = a\frac{dz}{du}$

From (2), $(3\frac{dz}{du})^{-1} + (3a\frac{dz}{du})^{-1} = z \Rightarrow \frac{1}{3}\frac{dz}{du} + \frac{1}{3a}\frac{dz}{du} = z$

$$\Rightarrow \frac{1}{3}\frac{du}{dz}(1 + \frac{1}{a}) = z$$

$$\Rightarrow zdz = \left(\frac{a+1}{3a}\right)du$$

On integrating, we get

$$\frac{z^2}{2} = \frac{a+1}{3a}u + c$$

$$\Rightarrow \frac{z^2}{2} = \frac{a+1}{3a}(X + aY) + c$$

Hence the general solution is

$$\frac{z^2}{2} = \frac{a+1}{3a}(x^3 + ay^3) + c$$

Type-VI: Equations of the type $f(x, pz^n) = g(y, qz^n)$

Method of solution:

The above form can be transformed to any of the three forms of by following substitution

put $Z = z^{n+1}$ if $n \neq -1$

and Z= logz if n=-1

Problems

1.Solve $(x+zp)^2+(y+zq)^2=1$

Sol:Given equation can be written as

$$(x+zp)^2=1-(y+zq)^2 \rightarrow (1)$$

This is in the form of $f(x, z^n p)=g(y, z^n q)$

Here n=1

Put $Z=z^{n+1}=z^{1+1}=z^2$

now, $$P=\frac{\partial Z}{\partial x}=\frac{\partial Z}{\partial z}\frac{\partial z}{\partial x}=2z.p$$

$$Q=\frac{\partial Z}{\partial y}=\frac{\partial Z}{\partial z}\frac{\partial z}{\partial y}=2z.q$$

Substituting in (1), we get

$$\left(\frac{P}{2}+x\right)^2 = 1-\left(\frac{Q}{2}+y\right)^2 = a^2(say)$$

$$\left(\frac{P}{2}+x\right)^2 = a^2, 1-\left(\frac{Q}{2}+y\right)^2 = a^2$$

$$P = 2(a-x), Q = 2[\sqrt{1-a^2}-y]$$

We know that,

$$dZ = Pdx + Qdy$$

$$dZ = 2(a-x)dx + 2[\sqrt{1-a^2}-y]dy$$

On integrating, we get

$$Z = 2(ax - \frac{x^2}{2}) + 2[\sqrt{1-a^2}\,y - \frac{y^2}{2}] + c$$

Hence the general solution is

$$z^2 = 2ax - x^2 + 2y\sqrt{1-a^2} - y^2 + c$$

	NON-LINEAR EQUATION	SOLUTION
Type 1	$f(p, q)=0$	$Z=ax+by+c$ $(p=a \text{ and } q=b)$
Type2	$f(z, p, q)=0$	$Z=\emptyset(x+ay)$
Type3	$f(x, p)=g(y, q)$	$dz=p\,dx+q\,dy$
Type4	$Z=px+qy+f(p, q)$	$Z=ax+by+f(a, b)$

Practice problems

1. Solve the PDE $z^2 = 1 + p^2 + q^2$

2. Solve the PDE $p^2 - q^2 = 1$

3. Solve the PDE $p(1+q) = qz$

4. Solve the PDE $pq = p + q$

Practice problems

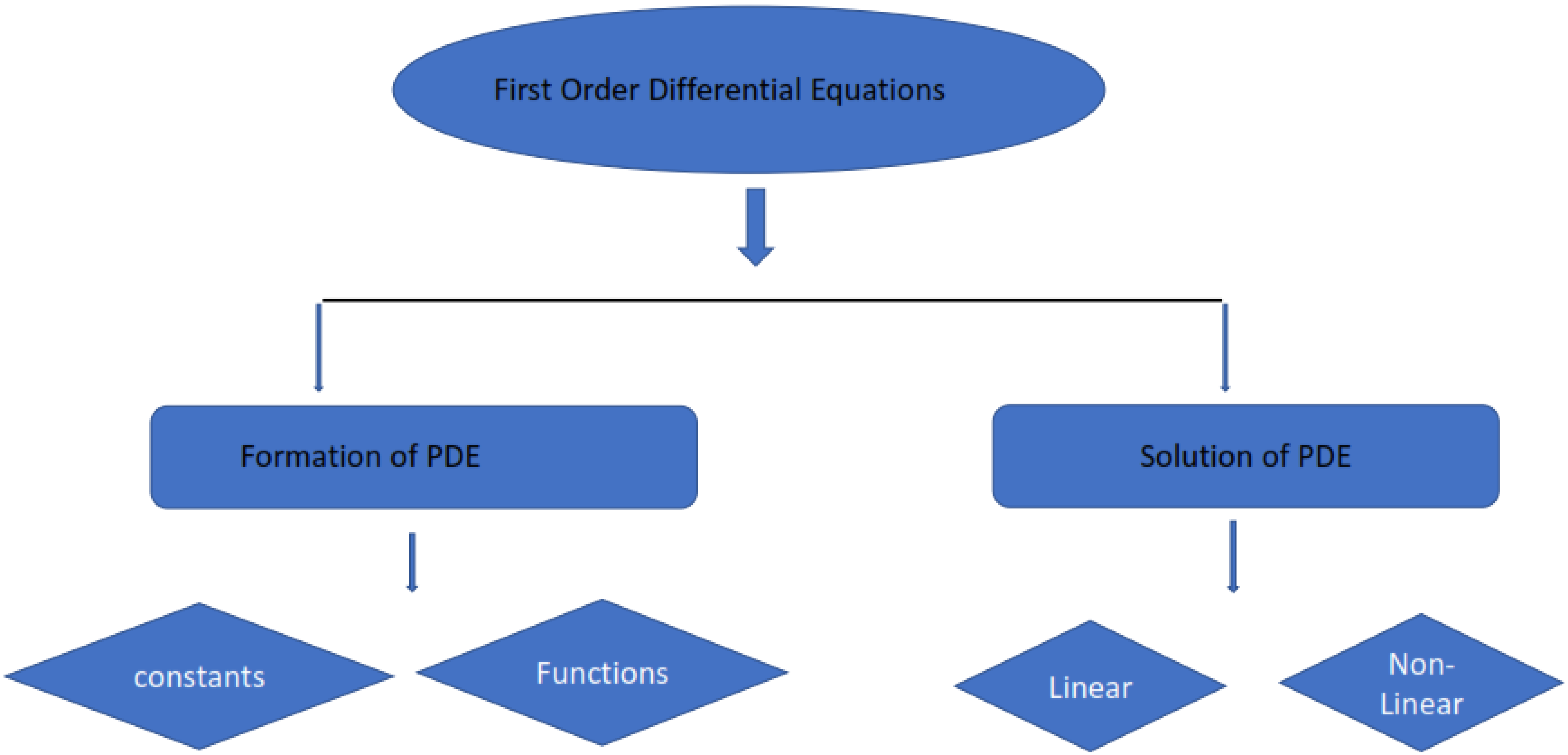
5. Solve the PDE $z = px + qy + \sqrt{p^2 + q^2 + 1}$

6. Solve the PDE $q^2 - p = y - x$

7. Solve the PDE $\dfrac{p}{x^2} + \dfrac{q}{y^2} = z$

8. Solve the PDE $z^2(p^2 + q^2) = x^2 + y^2$

9. Solve the PDE $z(p^2 - q^2) = x - y$


First Order Differential Equations
Formation of PDE
Solution of PDE
constants
Functions
Linear
Non-
Linear

Homogenous Linear PDE with constant coefficients

An equation of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + ... + a_n \frac{\partial^n z}{\partial y^n} = F(x,y) \rightarrow (1)$$

Where $a_0, a_1, a_2, ..., a_n$ are constants and F(x,y) is a function in x and y is called Homogenous Linear PDE with constant coefficients of order n.

Write $\frac{\partial}{\partial x} = D$ and $\frac{\partial}{\partial y} = D'$ then (1) can be written as

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + ... + a_n D'^n) z = F(x,y) \rightarrow (2)$$

$$\Rightarrow f(D, D') z = F(x,y) \rightarrow (2)$$

Solution of Homogenous Linear PDE with constant coefficients

The complete solution is given by

z=complementary function(C.F)+ Particular integral(P.I)

Where (a)complementary function(C.F) is the solution of the equation

$$f(D,D')z=0$$

(b) Particular integral(P.I) is the particular solution of $f(D,D')z=F(x,y)$

i.e., $\dfrac{1}{f(D,D')}F(x,y)$

WORKING PROCEDURE TO SOLVE THE EQUATION

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \dots + k^n \frac{\partial^n z}{\partial y^n} = F(x, y).$$

Its *symbolic form* is $(D^n + k_1 D^{n-1} D' + \dots + k_n D'^n)z = F(x, y)$

or briefly $f(D, D')z = F(x, y)$

Step I. To find the C.F.

(i) Write the A.E. by replacing D by 'm' and D' by '1':

i.e., $m^n + k_1 m^{n-1} + \dots + k_n = 0$ *and solve it for m.*

(ii) Write the C.F. as follows

Roots of A.E.	*C.F.*
1. $m_1, m_2, m_3 \dots$ (distinct roots)	$f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x) + \dots$
2. $m_1, m_1, m_3 \dots$ (two equal roots)	$f_1(y + m_1x) + xf_2(y + m_1x) + f_3(y + m_3x) + \dots$
3. $m_1, m_1, m_1 \dots$ (three equal roots)	$f_1(y + m_1x) + xf_2(y + m_1x) + x^2f_3(y + m_1x) + \dots$

To Find Particular Integral

Consider the symbolic form of the equation as

$$f(D, D')z = F(x, y) \qquad \ldots(1)$$

For this, Particular Integral $(\text{P. I.}) = \dfrac{1}{f(D, D')} F(x, y) \qquad \ldots(2)$

Case I: When $F(x, y) = e^{ax + by}$

$$\text{P.I.} = \frac{1}{f(D, D')} e^{ax+by} = \frac{1}{f(a, b)} e^{ax+by}, \quad f(a, b) \neq 0.$$

(*i.e.*, replace D by a, D' by b)

Case II: When $F(x, y) = \sin(ax + by)$ or $\cos(ax + by)$,

$$\text{P.I.} = \frac{1}{f(D, D')} \sin(ax + by)$$

$$= \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax + by),$$ Provided $f(-a^2, -ab, -b^2) \neq 0$.

(*i.e.*, replace $D^2 = -a^2$, $D'^2 = -b^2$, $DD' = -ab$)

Case (iii) When $F(x, y) = x^m y^n$

P. I. $= \dfrac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$

Expand $[f(D, D')]^{-1}$ in ascending powers of D or D' as follows and then operate on $x^m y^n$ term by term.

(a) If $m = n$, then expand in either power of D or D'.

(b) If $m < n$, then expand in power of D i.e., $\dfrac{D}{D'}$

(c) If $m > n$, then expand in power of D' i.e., $\dfrac{D'}{D}$

Step 6 : The complete solution of the given equation is $z = \text{C. F.} + \text{P. I.}$

lote : If in a homogeneous equation the R.H.S is zero then the C.F. gives the complete olution. In other words, if R. H. S. is zero, we need not find P. I.

Problems

1.Solve $\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial^3 u}{\partial x \partial y^2} - 2\frac{\partial^3 u}{\partial y^3} = 0$

Sol: The given equation can be written as

$(D^3 + 2D^2D' - DD'^2 - 2D'^3)u = 0$ where $\frac{\partial}{\partial x} = D$, $\frac{\partial}{\partial y} = D'$

put D=m and $D' = 1$then the auxiliary equation is given

$$f(m,1) = m^3 + 2m^2 - m - 2 = 0 \Rightarrow (m-1)(m+1)(m+2) = 0$$

$$\Rightarrow m = -1, 1, -2$$

$$u = f_1(y - x) + f_2(y + x) + f_3(y - 2x)$$

which is the required solution

2.Solve $\dfrac{\partial^4 z}{\partial x^4} - \dfrac{\partial^4 z}{\partial y^4} = 0$

Sol: The given equation can be written as

$$(D^4 - D'^4)z = 0 \quad \text{where} \quad \frac{\partial}{\partial x} = D, \frac{\partial}{\partial y} = D'$$

put D=m and $D' = 1$then the auxiliary equation is given

$$f(m,1) = m^4 - 1 = 0 \Rightarrow (m^2 - 1)(m^2 + 1) = 0$$

$$\Rightarrow m = -1, 1, -i, i$$

$$z = f_1(y + x) + f_2(y - x) + f_3(y + ix) + f_4(y - ix)$$

which is the required solution

3.Solve $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^2 z}{\partial x^2 \partial y} + 3\frac{\partial^3 z}{\partial x \partial y^2} - \frac{\partial^3 z}{\partial y^3} = 0$

Sol: The given equation can be written as

$(D^3 - 3D^2D' + 3DD'^2 - D'^3)z = 0$ where $\frac{\partial}{\partial x} = D, \frac{\partial}{\partial y} = D'$

put D=m and $D' = 1$then the auxiliary equation is given

$$f(m,1) = m^3 - 3m^2 + 3m - 1 = 0 \Rightarrow (m-1)^3 = 0$$

$$\Rightarrow m = 1,1,1$$

$$z = f_1(y+x) + xf_2(y+x) + x^2 f_3(y+x)$$

which is the required solution

4.Solve $(D^3-3D^2D'+4D'^3)z=e^{x+2y}$

Sol: The given equation is

$(D^3-3D^2D'+4D'^3)z=e^{x+2y}$ where $\frac{\partial}{\partial x}=D$, $\frac{\partial}{\partial y}=D'$

C.F: put D=m and $D'=1$ then the auxiliary equation is given

$$f(m,1)=m^3-3m^2+4=0 \Rightarrow (m-2)^2(m+1)=0$$

$$\Rightarrow m=2,2,-1$$

$$C.F = f_1(y+2x)+xf_2(y+2x)+f_3(y-x)$$

,

Now **P.I** = $\frac{1}{(D^3 - 3D^2D' + 4D'^3)} e^{x+2y}$

$$= \frac{1}{(1)^3 - 3(1)^2 2 + 4(2)^3} e^{x+2y}$$ [Put D=1, $D' = 2$]

$$= \frac{1}{1-6+32} e^{x+2y} = \frac{1}{27} e^{x+2y}$$

Hence the complete solution is given by

$$z = C.F + P.I = f_1(y+2x) + xf_2(y+2x) + f_3(y-x) + \frac{1}{27} e^{x+2y}$$

5.Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$

Sol: The given equation can be written as

$(D^2 - DD')z = \sin x \cos 2y$ where $\frac{\partial}{\partial x} = D$, $\frac{\partial}{\partial y} = D'$

C.F: put D=m and $D' = 1$ then the auxiliary equation is given

$$f(m,1) = m^2 - m = 0 \Rightarrow m(m-1) = 0$$
$$\Rightarrow m = 0,1$$

$$C.F = f_1(y) + f_2(y+x)$$

,

Now **P.I** = $\frac{1}{(D^2 - DD')}\sin x\cos 2y = \frac{1}{(D^2 - DD')}(\frac{1}{2}2\sin x\cos 2y)$

$$= \frac{1}{2}\frac{1}{(D^2 - DD')}[\sin(x+2y) + \sin(x-2y)]$$

$$= \frac{1}{2}[\frac{1}{-1-(-2)}\sin(x+2y) + \frac{1}{-1-(2)}\sin(x-2y)]$$

$$= \frac{1}{2}[\sin(x+2y) - \frac{1}{3}\sin(x-2y)] = \frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y)$$

Hence the complete solution is given by

$$z = C.F + P.I = f_1(y) + f_2(y+x)$$
$$+ \frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y)$$

6.Solve $(D^3 - 7D'^2 - 6D'^3)z = \sin(x+2y) + e^{2x+y}$

Sol: The given equation is

$$(D^3 - 7D'^2 - 6D'^3)z = \sin(x+2y) + e^{2x+y}$$

C.F: put D=m and $D' = 1$ then the auxiliary equation is given

$$f(m,1) = m^3 - 7m - 6 = 0 \Rightarrow (m+1)(m-3)(m+2) = 0$$

$$\Rightarrow m = -1, 3, -2$$

$$C.F = \phi_1(y-x) + \phi_2(y+3x) + \phi_3(y-2x)$$

$$\text{Now P. I.} = \frac{1}{D^3 - 7DD'^2 - 6D'^3}[\sin(x+2y) + e^{2x+y}]$$

$$= \frac{1}{D^3 - 7DD'^2 - 6D'^3}\sin(x+2y) + \frac{1}{D^3 - 7DD'^2 - 6D'^3}e^{2x+y}$$

$$= \frac{1}{D.D^2 - 7D.D'^2 - 6D'.D'^2}\sin(x+2y) + \frac{1}{(2)^3 - 7(2)(1)^2 - 6(1)^3}.e^{2x+y}$$

$$= \frac{1}{D(-1) - 7D(-4) - 6D'(-4)}\sin(x+2y) + \frac{1}{8-14-6}e^{2x+y}$$

$$= \frac{1}{-D + 28D + 24D'} \sin(x + 2y) - \frac{1}{12} e^{2x+y}$$

$$= \frac{1}{3(9D + 8D')} \sin(x + 2y) - \frac{1}{12} e^{2x+y}$$

$$= \frac{(9D - 8D')}{3(81D^2 - 64D'^2)} \sin(x + 2y) - \frac{1}{12} e^{2x+y}$$

[Multiplying numerator and denominator with conjugate term $(9D - 8D')$]

$$\therefore \quad \text{P. I.} = \frac{(9D - 8D')}{3[81(-1) - 64(-4)]} \sin(x + 2y) - \frac{1}{12} e^{2x+y}$$

$$= \frac{1}{525}(9D - 8D') \sin(x + 2y) - \frac{1}{12} e^{2x+y}$$

$$= \frac{1}{525}\left[9.\frac{\partial}{\partial x}\{\sin(x + 2y)\} - 8.\frac{\partial}{\partial y}\{\sin(x + 2y)\}\right] - \frac{1}{12} e^{2x+y}$$

$$= \frac{1}{525}[9\cos(x + 2y) - 16\cos(x + 2y)] - \frac{1}{12} e^{2x+y}$$

$$= \frac{-7}{525} \cos(x + 2y) - \frac{1}{12} e^{2x+y}$$

$$= \frac{-1}{75} \cos(x + 2y) - \frac{1}{12} e^{2x+y}$$

$\therefore$ **The complete solution is**

$$y = \text{C. F.} + \text{P. I.} = \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x) - \frac{1}{75}\cos(x + 2y) - \frac{1}{12} e^{2x+y}$$

7.Solve $(D^2 - 2DD')z = e^{2x} + x^3 y$

Sol: The given equation is

$$(D^2 - 2DD')z = e^{2x} + x^3 y$$

C.F: put D=m and $D' = 1$ then the auxiliary equation is given

$$f(m,1) = m^2 - 2m = 0 \Rightarrow m(m-2) = 0$$

$$\Rightarrow m = 0,2$$

$$C.F = \phi_1(y) + \phi_2(y + 2x)$$

,

$$\text{Now P. I.} = \frac{1}{D^2 - 2DD'}(e^{2x} + x^3 y) = \frac{1}{D^2 - 2DD'}e^{2x} + \frac{1}{D^2 - 2DD'}x^3 y$$

$$= \frac{e^{2x}}{4 - 2(0)} + \frac{1}{D^2\left(1 - \frac{2D'}{D}\right)}x^3 y = \frac{e^{2x}}{4} + \frac{1}{D^2}\left[1 - \frac{2D'}{D}\right]^{-1} x^3 y$$

$$= \frac{e^{2x}}{4} + \frac{1}{D^2}\left[1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \ldots\right]x^3 y$$

$$= \frac{e^{2x}}{4} + \frac{1}{D^2}\left[1 + \frac{2D'}{D}\right]x^3 y \quad \left[\because D'^2(x^3 y) = 0 = D'^3(x^3 y)\right]$$

$$= \frac{e^{2x}}{4} + \frac{1}{D^2}\left[x^3 y + \frac{2}{D}.\frac{\partial}{\partial y}(x^3 y)\right]$$

$$\therefore \text{ P. I. } = \frac{e^{2x}}{4} + \frac{1}{D^2}\left[x^3 y + \frac{2}{D}(x^3)\right] = \frac{e^{2x}}{4} + \frac{1}{D^2}(x^3 y) + \frac{2}{D^3}(x^3)$$

$$= \frac{e^{2x}}{4} + \frac{1}{D}\left[\int x^3 y\, dx\right] + \frac{2}{D^2}\left[\int x^3 dx\right] = \frac{e^{2x}}{4} + \frac{1}{D}\left[y.\frac{x^4}{4}\right] + \frac{2}{D^2}\left[\frac{x^4}{4}\right]$$

$$= \frac{e^{2x}}{4} + \frac{y}{4}\int x^4 dx + \frac{2}{4}.\frac{1}{D}\int x^4 dx = \frac{e^{2x}}{4} + \frac{y}{4}\left(\frac{x^5}{5}\right) + \frac{1}{2}.\frac{1}{D}\left(\frac{x^5}{5}\right)$$

$$= \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{1}{10}\int x^5 dx = \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{1}{10}\left(\frac{x^6}{6}\right)$$

$$= \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{x^6}{60}$$

$\therefore$ The complete solution of the equation is

$$y = \text{C. F.} + \text{P. I.} = \phi_1(y) + \phi_2(y+2x) + \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{x^6}{60}$$

Case of Failure :

Let $f(D,D')$ be the homogeneous function of degree n.

If on replacing D by 'a' and D' by 'b' in $f(D,D')$, we get $f(a,b)=0$, then in this case

Differentiate $f(D,D')$ partially w.r.t. D and multiply by x.

$$i.e., \text{ P. I. } = \frac{1}{f(D,D')}F(ax+by), \text{where } f(a,b)=0$$

$$= x.\frac{1}{\frac{\partial}{\partial D}[f(D,D')]}F(ax+by) \quad \text{or} \quad y.\frac{1}{\frac{\partial}{\partial D'}[f(D,D')]}F(ax+by)$$

If $f(a,b)=0$ again, then

$$\text{P. I. } = x^2.\frac{1}{\frac{\partial^2}{\partial D^2}[f(D,D')]}F(ax+by)$$

If $f(a,b)\neq 0$ then stop.

1.Solve $(4r+12s+9t)z = e^{3x-2y}$

Sol: we have $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial x}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y} \,\&\, t = \frac{\partial^2 z}{\partial y^2}$

The given equation is $4\frac{\partial^2 z}{\partial x^2} + 12\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = e^{3x-2y}$

$$\Rightarrow (4D^2 + 12DD' + 9D'^2)z = e^{3x-2y}$$

C.F: put D=m and $D' = 1$ then the auxiliary equation is given

$$f(m,1) = 4m^2 + 12m + 9 = 0 \Rightarrow (2m+3)^2 = 0$$

$$\Rightarrow m = \frac{-3}{2}, \frac{-3}{2}$$

$$C.F = f_1(y - \frac{3}{2}x) + xf_2(y - \frac{3}{2}x)$$

,

Now **P.I** $= \dfrac{1}{(4D^2 + 12DD' + 9D'^2)} e^{3x-2y}$

$$= \frac{1}{36 - 72 + 36} e^{3x-2y}$$ [Put D=3, $D' = -2$]

$$= \frac{1}{0} e^{3x-2y}$$ -----case of failure

$$\therefore P.I = x \frac{1}{\frac{\partial}{\partial D}(4D^2 + 12DD' + 9D'^2)} e^{3x-2y}$$

$$= x \frac{1}{(8D + 12D')} e^{3x-2y} = x \frac{1}{(24 - 24)} e^{3x-2y} = x \frac{1}{0} e^{3x-2y}$$

(Case of failure)

’

$$\therefore P.I = x^2 \frac{1}{\frac{\partial}{\partial D}(8D + 12D')} e^{3x-2y}$$

$$= x^2 \frac{1}{8} e^{3x-2y} = \frac{x^2}{8} e^{3x-2y}$$

Hence the complete solution is given by

$$z = C.F + P.I = f_1(y - \frac{3}{2}x) + xf_2(y - \frac{3}{2}x) + \frac{x^2}{2} e^{3x-2y}$$

5.Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = 3\sin(x+y)$

Sol: The given equation is

$$(D^3 + D^2D' - DD'^2 - D'^3)z = 3\sin(x+y)$$

C.F: put D=m and $D' = 1$ then the auxiliary equation is given

$$f(m,1) = m^3 + m^2 - m - 1 = 0 \Rightarrow (m^2 - 1)(m+1) = 0$$

$$\Rightarrow m = -1,1,-1 \Rightarrow m = -1,-1,1$$

$$C.F = f_1(y-x) + xf_2(y-x) + f_3(y+x)$$

,

Now **P.I** = $\frac{1}{(D^3+D^2D'-DD'-D'^3)}3\sin(x+y)$

$$=\frac{1}{-D-D'+D+D'}3\sin(x+y)$$ [Put $D^2=-1, D'^2=-1$]

$$=\frac{1}{0}3\sin(x+y)$$ -----case of failure

$$\therefore P.I = x\frac{1}{\frac{\partial}{\partial D}(D^3+D^2D'-DD'^2-D'^3)}3\sin(x+y)$$

$$= x\frac{1}{3D^2+2DD'-D'^2}3\sin(x+y)$$

,

$$= x\frac{1}{3(-1)+2(-1)-(-1)}3\sin(x+y)$$

$$= \frac{x}{-3-2+1}3\sin(x+y) = \frac{-3x}{4}\sin(x+y)$$

Hence the complete solution is given by

$$z = C.F + P.I = f_1(y-x) + xf_2(y-x) + f_3(y+x) - \frac{3x}{4}\sin(x+y)$$

,

General method

Case IV: When $F(x, y)$ is any function of x and y.

$$\text{P.I.} = \frac{1}{f(D, D')} F(x, y) = \frac{1}{(D - m_1 D')(D - m_2 D') \ldots} F(x, y)$$

and $\dfrac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\,dx$, where $c = y + mx$.

Solve $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 3\dfrac{\partial^2 z}{\partial y^2} = \sqrt{x+3y}$.

Solution: The given equation which is an homogenous linear partial differential equation of 2nd order can be written in its symbolic form as

$$(D^2 - 4DD' + 3D'^2)z = (x+3y)^{1/2} \quad \ldots(1)$$

Its auxiliary equation is as (Replace D by m & D' by 1)

$$(m^2 - 4m + 3) = 0 \quad \text{or} \quad m = 1, 3$$

whence $\quad$ C.F. $= \phi_1(y+x) + \phi_2(y+3x) \quad \ldots(2)$

For Particular Integral,

$$\text{P.I.} = \frac{1}{f(D, D')} F(x, y) = \frac{1}{(D-D')(D-3D')}(x+3y)^{1/2}$$

$$= \frac{1}{(D-3D')} \int_{D-D'} (x+3y)^{1/2} dx$$

$$= \frac{1}{(D-3D')} \int \left[x + 3(c_1 - x)\right]^{\frac{1}{2}} dx \qquad \text{since } y + x = c_1 \text{ for } m = 1$$

$$= \frac{1}{(D-3D')}\int (3c_1 - 2x)^{\frac{1}{2}}\, dx$$

$$= \frac{1}{(D-3D')}\left[\frac{(3c_1 - 2x)^{3/2}}{-2 \times \frac{3}{2}}\right]$$

$$= \frac{1}{(D-3D')}\left[-\frac{1}{3}(x+3y)^{3/2}\right], \quad \text{On replacing } c_1 = y + x$$

$$= -\frac{1}{3}\int_{D-3D'} (x+3y)^{3/2}\, dx$$

$$= -\frac{1}{3}\int (x + 3c_2 - 9x)^{3/2}\, dx, \quad \text{as } (y + 3x) = c_2 \text{ for } m = 3$$

$$= -\frac{1}{3}\,\frac{(3c_2 - 8x)^{5/2}}{-8 \times \frac{5}{2}}$$

$$= \frac{(3c_2 - 8x)^{5/2}}{60} = \frac{(x + 3y)^{5/2}}{60}, \qquad \text{replacing} \quad c_2 = y + 3x$$

Here, complete solution

$$z = \phi_1(y + x) + \phi_2(y + 3x) + \frac{(x + 3y)^{5/2}}{60}$$

Solve $(D^2 - DD' - 2D'^2)z = (y - 1)e^x$.

Solution: Corresponding A.E. is

$$(m^2 - m - 2) = 0 \quad i.e., \quad (m - 2)(m + 1) = 0$$

or $\quad m = 2, -1$

whence $\quad \text{C.F.} = \phi_1(y + 2x) + \phi_2(y - x)$

Now $\quad \text{P.I.} = \dfrac{1}{D^2 - DD' - 2D'^2}(y - 1)\,e^x$

$$= \frac{1}{(D-2D')(D+D')}(y-1)\,e^{x}$$

$$= \frac{1}{(D-2D')}\underset{D+D'}{\int}(y-1)\,e^{x}\,dx$$

Corresponding to the factor $D + D'$, $y = c_1 + x$

$$\Rightarrow \qquad \text{P.I.} = \frac{1}{(D-2D')}\int (c_1 + x - 1)\,e^{x}\,dx$$

$$= \frac{1}{(D-2D')}\int \left((c_1-1)\,e^{x} + x\,e^{x}\right)dx$$

$$= \frac{1}{D-2D'}\left[(c_1-1)\,e^{x} + (x-1)\,e^{x}\right]$$

$$= \frac{1}{D-2D'}\left[(y-2)\,e^{x}\right]; \quad \text{replacing, } c_1 = (y-x)$$

$$\text{P.I.} = \frac{1}{D - 2D'}(y-2)\,e^x\,dx$$

Expressing $(y - 2)$ in terms of x as $y + 2x = c_2$ corresponding to the factor $(D - 2D')$

$$\Rightarrow \qquad \text{P.I.} = \int (c_2 - 2x - 2)\,e^x\,dx = \int ((c_2 - 2) - 2x)\,e^x\,dx$$

$$= (c_2 - 2)\;e^x - 2\,(x - 1)\;e^x$$

$$\text{P.I.} = ye^x, \text{ on replacing } c_2 \text{ by } (y + 2x)$$

Therefore complete solution $z = \phi_1(y + 2x) + \phi_2(y - x) + ye^x$.

Solve $(r + s - 6t) = y\cos x$.

Solution: The given equation can be written as

$$(D^2 + DD' - 6D'^2)z = y\cos x$$

Corresponding A.E. is $(m^2 + m - 6) = 0$

$$\Rightarrow \qquad (m + 3)(m - 2) = 0 \quad i.e., \quad m = -3, +2$$

$$\text{C.F.} = \phi_1(y-3x) + \phi_2(y+2x)$$

Now $$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2} y\cos x$$

$$= \frac{1}{(D+3D')(D-2D')} y\cos x$$

$$= \frac{1}{(D+3D')} \int_{D-2D'} y\cos x\, dx$$

$$= \frac{1}{(D+3D')} \int (c_1 - 2x)\cos x\, dx, \qquad \text{Here } y + 2x = c_1$$

$$= \frac{1}{(D+3D')} [(c_1 - 2x)\sin x - 2\cos x] \qquad \text{replace } c_1 = y + 2x$$

$$= \frac{1}{(D+3D')} (y\sin x - 2\cos x)$$

$$= \int_{D+3D'} y \sin x \, dx - \int_{D+3D'} 2 \cos x \, dx$$

$$= \int_{D+3D'} (c_2 + 3x) \sin x \, dx - 2 \int_{D+3D'} \cos x \, dx \quad y - 3x = c_2$$

$$= -(c_2 + 3x) \cos x - \int 3 (-\cos x) \, dx - 2 \int \cos x \, dx$$

$$\text{P.I.} = -y \cos x + \int \cos x \, dx = -y \cos x + \sin x, \quad \text{replacing } (c_2 + 3x) \text{ by } y$$

$z = \phi_1(y - 3x) + \phi_2(y + 2x) - y \cos x + \sin x$ as complete solution.

Solve $4r - 4s + t = 16 \log(x + 2y)$.

Solution: The symbolic form of the above equation is

$$(4D^2 - 4DD' + D'^2)z = 16\log(x + 2y)$$

Corresponding A.E. is $4m^2 - 4m + 1 = 0$

i.e., $$(2m - 1)^2 = 0 \quad \text{or} \quad m = \frac{1}{2}, \frac{1}{2}$$

$\therefore$ $$\text{C.F.} = \phi_1(2y + x) + x\phi_2(2y + x)$$

$$\text{P.I.} = \frac{1}{4\left(D - \frac{1}{2}D'\right)^2} \cdot 16\log(x + 2y)$$

$$= 4 \cdot \frac{1}{D - \frac{1}{2}D'} \int \log 2c\, dx \quad \text{as } (2y + x) = c$$

$$= 4 \cdot \frac{1}{D - \frac{1}{2} D'} \cdot x \log (x + 2y)$$

$$= 4 \cdot \int x \log 2c \, dx$$

$$= 4 \cdot \frac{x^2}{2} \cdot \log 2c$$

$$= 2x^2 \log 2c(x + 2y)$$

Hence the complete solution,

$$z = \phi_1(y + 2x) + x\phi_2(y + 2x) + 2x^2 \log (x + 2y).$$